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IDENTIFICATION OF DIFFERENTIAL SYSTEMS WITH TIME-VARYING COEFFICIENTS

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PREFACE

In previous work on differential approximation by the present authors, linear time-invariant differential equations were determined that minimize least-square difference criterion functions, and successive approximations were obtained by means of quasilinearization techniques. In the present paper, the authors consider the analogous problem for approximating to a differential system which is suspected of having time-varying coefficients, and it is shown that the same method is easily extendible.

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SUMMARY

The authors consider here the problem of obtaining an optimal fit to observed data by determining successive approximating systems of linear differential equations with time-varying coefficients. This latter characteristic introduces little additional complexity in applying the differential approximation and quasilinearization methods used in the time-invariant case. Two examples are considered in order to demonstrate the method.

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1. INTRODUCTION

In previous work on differential-system identification using the techniques of differential approximation and quasilinearization (see, for example, Refs. [1-6]), it has been assumed that the system under consideration was of a time-invariant nature that gave rise to an output function $f(t)$ in the time-interval $[0,T]$ defined by the nonlinear differential equation

$$(1.1) \qquad f^{(L)} = G(f,f',\ldots,f^{(L-1)},t),$$

with initial conditions

$$(1.2) \qquad f^{(j)}(0) = C_j, \quad j = 0,1,\ldots,L-1.$$

Methods were then used to fit a linear differential approximating function $u(t)$, that is, one satisfying a linear differential equation of the type

$$(1.3) \qquad u_M + \sum_{j=1}^{M} b_j u_{M-j} = 0,$$

where u_j is the j-th derivative $u^{(j)}(t)$ of $u(t)$, with initial conditions

$$(1.4) \qquad u_j(0) = c_j, \quad j = 0,1,\ldots,M-1.$$

The b_j and c_j were chosen in such a way as to minimize a square-difference criterion function of the differences between the theoretical $u(t)$ and the observed $f(t)$. In the event of data observed only at R discrete points of time t_i, $0 \le t_1 \le t_2 \le \cdots \le t_R \le T$, the criterion function is

$$(1.5) \qquad S_A = \sum_{i=1}^{R} [u_0(t_i) - f(t_i)]^2,$$

while for continuously observed output the analogous function is

$$(1.6) \qquad S_B = \int_0^T [u_0(t) - f(t)]^2 dt.$$

Even though in this time-invariant case the b_j were considered as unknown constants which had to be determined, nevertheless in order to obtain them by successive approximation using quasilinearization, the device* was used of treating them as additional state variables, with differential equations

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*In fact it is partly because of this device that the extension to time-varying systems is simple, since what we are doing is to go from artificially varying state variables or coefficients to actually varying coefficients.

(1.7) $\quad \dot{b}_j = 0.$

It will now be shown in some simple time-varying cases that these methods are easily extendible. We consider two examples involving feedback, which are related to differential systems occurring in models of such biological systems as the respiratory chemostat, and its blood and gas flow, in which the time-varying nature of the systems have up to now had to be ignored.

2. MODEL I

Suppose that $f(t)$ is defined by the "first-order" linear differential equation

(2.1) $$f'(t) = a(t)f(t) + b,$$

where the coefficient $a(t)$ is itself subject to the feedback equation

(2.2) $$a(t) = b_0 + b_1 f(t) + b_2 f_1(t) + \cdots + b_{k+1} f_k(t),$$

where $f_j(t) = f^{(j)}(t)$, the j-th derivative of $f(t)$.

Or alternatively, in the interval $[0,T]$ we wish to determine a function $u(t)$, defined by constants $b, b_0, b_1, \ldots, b_{k+1}$, by initial conditions

(2.3) $$u_j(0) = c_j, \quad j = 0,1,\ldots,k-1,$$

and by differential equations

$$(2.4) \qquad u_1(t) = a(t)u(t) + b,$$

where

$$(2.5) \qquad a(t) = b_0 + b_1u(t) + b_2u_1(t) + \cdots + b_{k+1}u_k(t),$$

that minimizes S_A (Eq. (1.5)) or S_B (Eq. (1.6)).

2.1. Method of Solution

Omitting the implied time variable t, we first obtain from equations (2.4) and (2.5)

$$(2.6) \qquad b_{k+1}u_k = \frac{u_1 - b}{u} - b_0 - b_1u - \cdots - b_ku_{k-1}.$$

[Continuity of the first k derivatives of $u(t)$ will be assumed so that, for example, when $u = 0$, the first term on the right-hand side of equation (2.6) is considered to be $\lim_{u\to 0} \frac{u_1 - b}{u}$.]

From equation (2.6), if we now use the device of assuming $b, b_0, \ldots, b_{k+1}$ to be state variables, we arrive at the system of differential equations

$$
(2.7)\qquad \begin{cases}
\dot{u}_0 = u_1 \\
\dot{u}_1 = u_2 \\
\vdots \\
\dot{u}_{k-2} = u_{k-1} \\
\dot{u}_{k-1} = \dfrac{1}{b_{k+1}}\left[\dfrac{u_1 - b}{u} - b_0 - b_1 u - \cdots - b_k u_{k-1}\right] \\
\dot{b} = 0 \\
\dot{b}_0 = 0 \\
\vdots \\
b_{k+1} = 0.
\end{cases}
$$

If we now put

$$
(2.8)\qquad \begin{cases}
u_0 = x_0 \\
\vdots \\
u_{k-1} = x_{k-1} \\
b = x_k \\
b_0 = x_{k+1} \\
\vdots \\
b_{k+1} = x_{2k+2},
\end{cases}
$$

then the nonlinear system of differential equations in the variable $\underline{x} = (x_0, x_1, \ldots, x_{2k+2})$ is given by

$$
(2.9)\qquad \dot{\underline{x}} = \underline{g}(\underline{x}),
$$

or

$$(2.10)\qquad \dot{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k-1} \\ \dfrac{1}{x_{2k+2}}\left[\dfrac{x_1 - x_k}{x_0} - x_{k+1} - x_{k+2}x_0 - \cdots - x_{2k+1}x_{k-1}\right] \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

As in [6], we now quasilinearize, obtaining successive solutions $\underline{x}^{(1)},\ldots,\underline{x}^{(n-1)},\underline{x}^{(n)},\ldots,$ the set of linear differential equations at the n-th stage being

$$(2.11)\qquad \dot{x}_j^{(n)} = g_j(\underline{x}^{(n-1)}) + \sum_{m=0}^{2k+2} \frac{\partial g_j(\underline{x}^{(n-1)})}{\partial x_m}\left(x_m^{(n)} - x_m^{(n-1)}\right).$$

For notational convenience in what follows, we now put $\underline{x}^{(n)} = \underline{y}$, and $\underline{x}^{(n-1)} = \underline{x}$, so that the linear system (2.11) for $\underline{y}$ becomes

$$(2.12)\qquad \dot{y}_j = g_j(\underline{x}) + \sum_{m=0}^{2k+2} \frac{\partial g_j(\underline{x})}{\partial x_m}(y_m - x_m).$$

From equation (2.10),

$$(2.13)\qquad \frac{\partial g_j}{\partial x_i} = \begin{cases} \delta_{ij} & \text{for } j = 0,\ldots,k-2,\ i = 0,\ldots,2k+2, \\ 0 & \text{for } j = k,\ldots,2k+2,\ i = 0,\ldots,2k+2, \end{cases}$$

and

$$
(2.14)\quad \left\{
\begin{aligned}
\frac{\partial g_{k-1}}{\partial x_0} &= -\frac{(x_1 - x_k)}{x_{2k+2}x_0^2} - \frac{x_{k+2}}{x_{2k+2}} \\
\frac{\partial g_{k-1}}{\partial x_1} &= \frac{1}{x_{2k+2}x_0} - \frac{x_{k+3}}{x_{2k+2}} \\
\frac{\partial g_{k-1}}{\partial x_2} &= \frac{x_{k+4}}{x_{2k+2}} \\
&\vdots \\
\frac{\partial g_{k-1}}{\partial x_{k-1}} &= -\frac{x_{2k+1}}{x_{2k+2}} \\
\frac{\partial g_{k-1}}{\partial x_k} &= -\frac{1}{x_{2k+2}x_0} \\
\frac{\partial g_{k-1}}{\partial x_{k+1}} &= -\frac{1}{x_{2k+2}} \\
\frac{\partial g_{k-1}}{\partial x_{k+2}} &= -\frac{x_0}{x_{2k+2}} \\
&\vdots \\
\frac{\partial g_{k-1}}{\partial x_{2k+1}} &= -\frac{x_{k-2}}{x_{2k+2}} \\
\frac{\partial g_{k-1}}{\partial x_{2k+2}} &= -\frac{1}{x_{2k+2}^2}\left[\frac{x_1 - x_k}{x_0} - x_{k+1} - \cdots - x_{2k+1}x_{k-2}\right].
\end{aligned}
\right.
$$

Hence equations (2.12) become

$$(2.15) \qquad \dot{\underline{y}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k-1} \\ h_1(\underline{y},\underline{x}) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where, after some manipulation, $h_1(\underline{y},\underline{x})$ is given by

$$(2.16) \qquad x_{2k+2}h_1(\underline{y},\underline{x}) = -\sum_{j=0}^{k-1}(y_j x_{k+2+j} + x_j y_{k+2+j}) + \frac{2(x_1 - x_k)}{x_0} - y_0\frac{(x_1 - x_k)}{x_0^2} + \frac{(y_1 - y_k)}{x_0} - y_{k+1} - x_{k+1} - \frac{y_{2k+2}}{x_{2k+2}}\left[\frac{x_1 - x_k}{x_0} - x_{k+1} - x_{k+2}x_0 - \cdots - x_{2k+1}x_{k-1}\right].$$

As in (1.5) and (1.6), at each stage we determine $(2k + 3)$ independent homogeneous solutions $\underline{y}_{hj}$ and one particular solution $\underline{y}_p$ for equations (2.15), and find the linear combination

$$(2.17)\qquad \underline{y} = \underline{y}_p + \sum_{j=0}^{2k+2} d_j \underline{y}_{hj},$$

with $\{d_j\}$ such that S_A or S_B is minimized. The procedure is precisely as in the time-invariant case ([5], [6]), to which the reader is referred.

After determining $\underline{y}$ ($= \underline{x}^{(n)}$), we then use it to find $\underline{x}^{(n+1)}$ in the same way, and so on.

(As in [5], [6], the $\underline{y}_{hj}$ and $\underline{y}_p$ are found with respective initial conditions

$$(2.18)\qquad (\underline{y}_{h0}(0)\underline{y}_{h1}(0)\cdots\underline{y}_{h,2k+2}(0)) = \underline{I}_{2k+3}$$

and

$$(2.19)\qquad \underline{y}_p(0) = \underline{0},$$

where $\underline{I}_{2k+3}$ is the $(2k + 3) \times (2k + 3)$ identity matrix.)

3. MODEL II

We now go on to consider a second model which, although conceptually rather more complicated than Model I, nevertheless introduces no additional difficulties and is handled in much the same manner.

Suppose now that instead of equations (2.4) and (2.5), we have

$$(3.1)\qquad u_1(t) = a(t)u(t) + b,$$

as before, and

$$(3.2)\qquad \dot{a}(t) = ca + b_0 + b_1 u + b_2 u_1 + \cdots + b_{k+1} u_k .$$

Then u_k may be expressed as

$$(3.3)\qquad u_k = \frac{1}{b_{k+1}}\left[\frac{u_2 - au_1}{u} - ca - b_0 - b_1 u - \cdots - b_k u_{k-1}\right].$$

But $a = (u_1 - b)/u$, so that

$$(3.4)\qquad u_k = \frac{1}{b_{k+1}}\left[\frac{uu_2 - u_1(u_1 - b)}{u^2} - \frac{c(u_1 - b)}{u} - b_0 - b_1 u - \cdots - b_k u_{k-1}\right].$$

Hence, putting

$$(3.5)\qquad \begin{cases} u_0 = x_0 \\ \vdots \\ u_{k-1} = x_{k-1} \\ b = x_k \\ b_0 = x_{k+1} \\ \vdots \\ b_{k+1} = x_{2k+2} \\ c = x_{2k+3}, \end{cases}$$

we obtain, as for Model I, the nonlinear system of differential equations

$$(3.6)\qquad \dot{\underline{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_{k-1} \\ \dfrac{1}{x_{2k+2}}\left[\dfrac{x_0x_2 - x_1(x_1-x_k)}{x_0^2} - \dfrac{x_{2k+3}(x_1-x_k)}{x_0} - x_{k+1}\right. \\ \left. - x_{k+2}x_0 - \cdots - x_{2k+1}x_{k-1}\right] \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where now, of course, $\underline{x} = (x_0, x_1, \ldots, x_{2k+3})$.

The iterative systems of quasilinear differential equations are obtained precisely as before, giving equations similar to equations (2.15) with $h_1(\underline{y},\underline{x})$ being replaced by $h_2(\underline{y},\underline{x})$, where

(3.7)

$$
\begin{aligned}
& x_{2k+2}h_2(\underline{y},\underline{x}) \\
&= -\sum_{j=0}^{k-1}(y_j x_{k+2+j} + x_j y_{k+2+j}) - y_{k+1} + \frac{y_2}{x_0} \\
&+ \frac{y_0}{x_0^3}\left[- x_2 x_0 + 2x_1^2 - 2x_1 x_k + x_0 x_1 x_{2k+3} - x_0 x_k x_{2k+3}\right] \\
&- \frac{y_1}{x_0^2}\left[2x_1 + x_k + x_0 x_{2k+3}\right] + \frac{y_k}{x_0^2}\left[x_1 + x_0 x_{2k+3}\right] \\
&- \frac{y_{2k+2}}{x_{2k+2}}\left[\frac{x_2 - x_0 - x_1^2 + x_1 x_k - x_0 x_{2k+3}(x_1 - x_k)}{x_0^2}\right. \\
&\qquad\qquad \left. - x_{k+1} - x_{k+2}x_0 - \cdots - x_{2k+1}x_{k-1}\right] \\
&- \frac{y_{2k+3}}{x_0}(x_1 - x_k) + \frac{2}{x_0^2}(x_1 x_k - x_1^2 + x_0 x_2) \\
&- \frac{x_{2k+3}}{x_0}(x_1 - x_k) - x_{k+1}.
\end{aligned}
$$

From here onwards, we proceed as before.

4. CONCLUSIONS

It becomes evident from examination of the models considered above that the techniques of differential approximation and quasilinearization applied previously to time-invariant system identification may also be applied widely to time-varying systems.

In further work, we shall investigate the extension of these methods to differential-difference systems.

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